

Hyperbolic Space Has Strong Negative Type

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Abstract. It is known that hyperbolic spaces have strict negative type, a condition on the distances of any finite subset of points. We show that they have strong negative type, a condition on every probability distribution of points (with integrable distance to a fixed point). This implies that the function of expected distances to points determines the probability measure uniquely. It also implies that the distance covariance test for stochastic independence, introduced by Székely, Rizzo and Bakirov, is consistent against all alternatives in hyperbolic spaces. We prove this by showing an analogue of the Cramér-Wold device.

§1. Introduction.

Let (X, d) be a metric space. One says that (X, d) has *negative type* if for all $n \geq 1$ and all lists of n red points x_i and n blue points x'_i in X , the sum $2 \sum_{i,j} d(x_i, x'_j)$ of the distances between the $2n^2$ ordered pairs of points of opposite color is at least the sum $\sum_{i,j} (d(x_i, x_j) + d(x'_i, x'_j))$ of the distances between the $2n^2$ ordered pairs of points of the same color. It is not obvious that euclidean space has this property, but it is well known. By considering repetitions of x_i and taking limits, we arrive at a superficially more general property: For all $n \geq 1$, $x_1, \dots, x_n \in X$, and $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ with $\sum_{i=1}^n \alpha_i = 0$, we have

$$\sum_{i,j \leq n} \alpha_i \alpha_j d(x_i, x_j) \leq 0. \quad (1.1)$$

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We say that (X, d) has **strict negative type** if, for every n and all n -tuples of distinct points x_1, \dots, x_n , equality holds in (1.1) only when $\alpha_i = 0$ for all i . Again, euclidean spaces have strict negative type. A simple example of a metric space of non-strict negative type is ℓ^1 on a 2-point space, i.e., \mathbb{R}^2 with the ℓ^1 -metric.

A (Borel) probability measure μ on X has **finite first moment** if $\int d(o, x) d\mu(x) < \infty$ for some (hence all) $o \in X$; write $P_1(X, d)$ for the set of such probability measures. Suppose that $\mu_1, \mu_2 \in P_1(X, d)$. By approximating μ_i by probability measures of finite support, we obtain a yet more general property, namely, that when X has negative type,

$$\int d(x_1, x_2) d(\mu_1 - \mu_2)^2(x_1, x_2) \leq 0. \quad (1.2)$$

We say that (X, d) has **strong negative type** if it has negative type and equality holds in (1.2) only when $\mu_1 = \mu_2$. See Lyons (2013) for an example of a (countable) metric space of strict but not strong negative type. The notion of strong negative type was first defined by Zinger, Kakosyan, and Klebanov (1992). Lyons (2013) used it to show that a metric space X has strong negative type iff the theory of distance covariance holds in X just as in euclidean spaces, as introduced by Székely, Rizzo, and Bakirov (2007). Lyons (2013) noted that if (X, d) has negative type, then (X, d^r) has strong negative type when $0 < r < 1$.

Define

$$a_\mu(x) := \int d(x, x') d\mu(x')$$

for $x \in X$ and $\mu \in P_1(X, d)$. Lyons (2013) remarked that if (X, d) has negative type, then the map $\alpha: \mu \mapsto a_\mu$ is injective on $\mu \in P_1(X)$ iff X has strong negative type. (There are also metric spaces not of negative type for which α is injective.)

The concept of negative type is old, but has enjoyed a resurgence of interest recently due to its uses in theoretical computer science, where embeddings of metric spaces, such as graphs, play a useful role in algorithms; see, e.g., Naor (2010) and Deza and Laurent (1997). A list of metric spaces of negative type appears as Theorem 3.6 of Meckes (2013); in particular, this includes all L^p spaces for $1 \leq p \leq 2$. On the other hand, \mathbb{R}^n with the ℓ^p -metric

is not of negative type whenever $3 \leq n \leq \infty$ and $2 < p \leq \infty$, as proved by Dor (1976) combined with Theorem 2 of Bretagnolle, Dacunha-Castelle, and Krivine (1965/1966); see Koldobsky and Lonke (1999) for an extension to spaces that include some Orlicz spaces, among others. Schoenberg (1937, 1938) showed that X is of negative type iff there is a Hilbert space H and a map $\phi: X \rightarrow H$ such that $\forall x, x' \in X$ $d(x, x') = \|\phi(x) - \phi(x')\|^2$.

That real and complex hyperbolic spaces \mathbb{H}^n have negative type was shown by Gangolli (1967), Sec. 4, and was made explicit by Faraut and Harzaallah (1974), Corollary 7.4; that they have strict negative type was shown by Hjorth, Kokkendorff, and Markvorsen (2002). (The proof of those last authors has some minor errors that are easily corrected.) We extend this as follows:

Theorem. *For all $n \geq 1$, real hyperbolic space $\mathbb{H}_{\mathbb{R}}^n$ of dimension n has strong negative type.*

It is open whether complex hyperbolic spaces $\mathbb{H}_{\mathbb{C}}^n$ have strong negative type, which would imply our theorem. More generally, it is open whether all Cartan-Hadamard manifolds have strong negative type, but Hjorth, Kokkendorff, and Markvorsen (2002) showed that those that have negative type have strict negative type. It is known that Cartan-Hadamard surfaces have negative type: see Chalopin, Chepoi, and Naves (2015).

§2. Proof of the Theorem.

Fix $o \in \mathbb{H}_{\mathbb{R}}^n$. Let σ be the (infinite) Borel measure on geodesic closed half-spaces $S \subset \mathbb{H}_{\mathbb{R}}^n$ that is invariant under isometries, normalized so that

$$\sigma(\{o \in S, x \notin S\}) = d(o, x)/2; \quad (2.1)$$

see Robertson (1998). Now let $\phi(x)$ be the function $S \mapsto \mathbf{1}_S(o) - \mathbf{1}_S(x)$ in $L^2(\sigma)$. It clearly satisfies Schoenberg's condition that $d(x, y) = \|\phi(x) - \phi(y)\|^2$. We call this the **Crofton embedding**, as Crofton (1868) was the first to give a formula for the distance of points in the plane in terms of lines intersecting the segment joining them. Thus, $\mathbb{H}_{\mathbb{R}}^n$ has negative type. In fact, we shall not use Schoenberg's theorem, even though this half is easy.

Instead, note that for $\mu_1, \mu_2 \in P_1(\mathbb{H}_{\mathbb{R}}^n)$, we have

$$\int d(x_1, x_2) d(\mu_1 - \mu_2)^2(x_1, x_2) = \int \int |\mathbf{1}_S(x_1) - \mathbf{1}_S(x_2)|^2 d(\mu_1 - \mu_2)^2(x_1, x_2) d\sigma(S).$$

Expanding the square and using the facts that

$$\int \mathbf{1}_S(x) d\nu^2(x, y) = \nu(S)\nu(X)$$

and

$$\int \mathbf{1}_S(x)\mathbf{1}_S(y) d\nu^2(x, y) = \nu(S)^2,$$

we obtain that

$$\int d(x_1, x_2) d(\mu_1 - \mu_2)^2(x_1, x_2) = -2 \int (\mu_1(S) - \mu_2(S))^2 d\sigma(S).$$

This clearly proves negative type; also, it is easy to prove strict negative type from this, using the fact that every finite set has a point that is in a half-space that does not contain any other point of the set. In order to prove strong negative type, it clearly suffices to show that if $\mu_1(S) = \mu_2(S)$ for σ -a.e. S and $\mu_1, \mu_2 \in P_1(\mathbb{H}_{\mathbb{R}}^n)$, then $\mu_1 = \mu_2$. Consider the Klein model of $\mathbb{H}_{\mathbb{R}}^n$ in which the space is the open unit ball of \mathbb{R}^n and in which geodesics are euclidean straight lines, whence hyperbolic half-spaces are the intersections of euclidean half-spaces with the unit ball. Every probability measure in the Klein model thus is a probability measure on \mathbb{R}^n that happens to be carried by the unit ball. The Cramér-Wold device (pp. 382–3 of Billingsley (1995)) now provides the desired conclusion. (The usual statement of the device is that if μ_1 and μ_2 are probability measures on euclidean space \mathbb{R}^n that satisfy $\mu_1(S) = \mu_2(S)$ for *all* half-spaces S , then $\mu_1 = \mu_2$. Its proof extends easily to the weaker hypothesis that $\mu_1(S) = \mu_2(S)$ for all S of the form $S = \{x \in \mathbb{R}^n; x \cdot t \leq \alpha\}$ for a set B of pairs $(t, \alpha) \in \mathbb{R}^n \times \mathbb{R}$ with the projection πB of B to \mathbb{R}^n being dense in \mathbb{R}^n and for each $t \in \pi B$, the set $\{\alpha; (t, \alpha) \in B\}$ being dense in \mathbb{R} . Alternatively, we may appeal to the fact that \mathbb{R}^n has strong negative type for our desired conclusion.) ■

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